Bernoulli actions of type III and L^2-cohomology

(joint work with Jonas Wahl)

Workshop L^2-invariants

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Bernoulli actions

Bernoulli actions of a countable group G

For any standard probability space (X_0, μ_0), consider

$$G \curvearrowright (X_0, \mu_0)^G = \prod_{g \in G} (X_0, \mu_0) \text{ given by } (g \cdot x)_h = x_{g^{-1}h}.$$

- $(G = \mathbb{Z})$ Kolmogorov-Sinai : entropy of μ_0 is a conjugacy invariant.
- $(G = \mathbb{Z})$ Ornstein : entropy is a complete invariant.
- Bowen : beyond amenable groups, sofic groups.
- Popa : orbit equivalence rigidity, von Neumann algebra rigidity.

$\rightsquigarrow$ What can be said about $G \curvearrowright \prod_{g \in G} (X_0, \mu_g)$?

Main motivation: produce interesting families of type III group actions.

Group actions of type III

- The classical Bernoulli action $G \curvearrowright (X, \mu) = (X_0, \mu_0)^G$
  - is ergodic,
  - preserves the probability measure μ.
- An action $G \curvearrowright (X, \mu)$ is called **non-singular** if $\mu(g \cdot \mathcal{U}) = 0$ whenever $\mu(\mathcal{U}) = 0$ and $g \in G$.
- Write $\mathcal{U} \sim \mathcal{V}$ if there exists a measurable bijection $\Delta : \mathcal{U} \to \mathcal{V}$ with $\Delta(x) \in G \cdot x$ for a.e. $x \in \mathcal{U}$.
- A nonsingular ergodic $G \curvearrowright (X, \mu)$ is of **type III** if $\mathcal{U} \sim \mathcal{V}$ for all non-negligible $\mathcal{U}, \mathcal{V} \subset X$.
  - There is no G-invariant measure in the measure class of μ.
  - The **Radon-Nikodym derivative** $d(g \cdot \mu)/d\mu$ must be sufficiently wild.

Group actions of type III_1

Let $G \curvearrowright (X, \mu)$ be a nonsingular group action.

- Write $\omega(g, x) = \frac{d(g \cdot \mu)}{d\mu}(x)$, the Radon-Nikodym 1-cocycle.
- The action $G \curvearrowright X \times \mathbb{R}$ given by $g \cdot (x, s) = (g \cdot x, s + \log(\omega(g, x)))$ preserves the (infinite) measure $\mu \times e^{-s} ds$.
- This is called the **Maharam extension.** It is the ergodic analogue of the **Connes-Takesaki continuous core** for von Neumann algebras.

$\rightsquigarrow$ An ergodic nonsingular action $G \curvearrowright (X, \mu)$ is of **type $\mathbf{III_1}$** if its Maharam extension remains ergodic.

$\rightsquigarrow$ Digression: the action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^G$ is the **flow of weights.**

$\rightsquigarrow$ $G \curvearrowright (X, \mu)$ is of type III iff this flow is not just $\mathbb{R} \curvearrowright \mathbb{R}$.

Bernoulli actions of type III

Consider $G \curvearrowright (X, \mu) = \prod_{g \in G} (X_0, \mu_g)$ given by $(g \cdot x)_h = x_{g^{-1}h}$.

1. All μ_g are equal : type II_1, ergodic, probability measure preserving.
2. **Interesting gray zone :** when is $G \curvearrowright (X, \mu)$ of type III, or type III_1 ?
3. The μ_g are quite different : type I, the action is **dissipative**, meaning that $X = \bigsqcup_{g \in G} g \cdot \mathcal{U}$ up to measure zero.
4. The μ_g are very different : the action is singular.

Kakutani's criterion

- The action $G \curvearrowright \prod_{g \in G} (X_0, \mu_g)$ is nonsingular if and only if

  for every $g \in G$, we have $\sum_{h \in G} d(\mu_{gh}, \mu_h)^2 < \infty$.

- Take $X_0 = \{0, 1\}$ with $\mu_g(0) = F(g)$ and $\mu_g(1) = 1 - F(g)$.
  Assume that $\delta \leq F(g) \leq 1 - \delta$ for all $g \in G$.

  Then, the action is nonsingular if and only if

  $\sum_{h \in G} |F(gh) - F(h)|^2 < \infty$ for all $g \in G$.

⇝ Then $c : G \to \ell^2(G) : c_g(h) = F(g^{-1}h) - F(h)$ is a **1-cocycle** for the left regular representation,

meaning that $c_{gh} = c_g + \lambda_g c_h$.

L^2-cohomology and L^2-Betti numbers

Consider the L^2-cohomology $H^1(G, \ell^2(G))$: the space of 1-cocycles divided by the 1-coboundaries, i.e. the 1-cocycles of the form $c_g = \xi - \lambda_g \xi$ for some $\xi \in \ell^2(G)$.

- The right action of G commutes with the left action λ_g.
- In this way, $H^1(G, \ell^2(G))$ becomes a right $L(G)$-module, where $L(G)$ is the group von Neumann algebra of G.
- Then, $\beta_1^{(2)}(G)$ is the Murray-von Neumann dimension of this module.
- (Cheeger-Gromov) When G is amenable, we have $\beta_1^{(2)}(G) = 0$, although for infinite amenable G, we have $H^1(G, \ell^2(G)) \neq \{0\}$.
- For nonamenable groups G, we have $\beta_1^{(2)}(G) = 0$ if and only if $H^1(G, \ell^2(G)) = \{0\}$.

An easy no-go theorem

Theorem (V-Wahl, 2017)

If $H^1(G, \ell^2(G)) = \{0\}$, there are **no nonsingular Bernoulli actions of type III.** More precisely,

every nonsingular Bernoulli action of G is the sum of a classical, probability measure preserving Bernoulli action and a dissipative Bernoulli action.

- The groups with $H^1(G, \ell^2(G)) = \{0\}$ are precisely the nonamenable groups with $\beta_1^{(2)}(G) = 0$.
- Large classes of nonamenable groups have $\beta_1^{(2)}(G) = 0$:
  - property (T) groups,
  - groups that admit an infinite, amenable, normal subgroup,
  - direct products of infinite groups.

What if $H^1(G, \ell^2(G)) \neq \{0\}$?

This is very delicate !

Even for the case $G = \mathbb{Z}$.

- (Hamachi, 1981)

  The group $G = \mathbb{Z}$ admits a nonsingular Bernoulli action of type **III**

- (Kosloff, 2009)

  The group $G = \mathbb{Z}$ admits a nonsingular Bernoulli action of type $\mathbf{III_1}$

⤳ In both cases: no explicit construction.

What if $H^1(G, \ell^2(G)) \neq \{0\}$?

Theorem (V-Wahl, 2017)

Take $\mathbb{Z} \curvearrowright^T \prod_{n \in \mathbb{Z}} (\{0,1\}, \mu_n)$.

For $n \leq 0$, we take $\mu_n(0) = 1/2$. For $n \geq 1$, we take respectively

- (folklore) $\mu_n(0) = p$ with $1/2 < p < 1$: the action T is dissipative,
- $\mu_n(0) = \dfrac{1}{2} + \dfrac{1}{6\sqrt{n}}$: the action T is ergodic and of type III_1, but the 73-fold diagonal product $T \times \cdots \times T$ is dissipative,
- $\mu_n(0) = \dfrac{1}{2} + \dfrac{1}{\sqrt{5 + n \log n}}$: the action T and all its powers are ergodic and of type III_1,
- (Kakutani) $\sum_n (\mu_n(0) - 1/2)^2 < \infty$: type II_1, classical Bernoulli action.

Main results of V-Wahl (2017)

Theorem A

All infinite amenable groups G admit nonsingular, ergodic Bernoulli actions of type III_1.

For most of these, we can take $X_0 = \{0, 1\}$.

Theorem B

Most countable groups G with $\beta_1^{(2)}(G) > 0$ admit nonsingular, ergodic Bernoulli actions of type III_1.

We can prove Theorem B in the following cases:

1. when G admits an infinite subgroup $G_0 < G$ with $\beta_1^{(2)}(G_0) < \beta_1^{(2)}(G)$,
2. when G admits a subgroup $G_0 < G$ with $\beta_1^{(2)}(G)^{-1} \leq [G : G_0] < \infty$.

Group theoretic observations

Assume that $\beta_1^{(2)}(G) > 0$.

1. The existence of an infinite subgroup $G_0 < G$ with $\beta_1^{(2)}(G_0) < \beta_1^{(2)}(G)$ is automatic in the following cases.

   - When G has at least one element of infinite order.
   - When G admits an infinite amenable subgroup.
   - When $\beta_1^{(2)}(G) \geq 1$, relying on the following remarkable result.

   If Γ is any infinite group and if every pair of elements $a, b \in \Gamma$ generates a finite subgroup, then Γ contains an infinite abelian subgroup !

   All known proofs invoke the Feit-Thompson odd order theorem.

2. The existence of a subgroup $G_0 < G$ with $\beta_1^{(2)}(G)^{-1} \leq [G : G_0] < \infty$ is automatic if G is residually finite.

The easiest case of Theorem B

Assume that $G_0 < G$ is an infinite subgroup with $\beta_1^{(2)}(G_0) < \beta_1^{(2)}(G)$.

- The restriction map $H^1(G, \ell^2(G)) \to H^1(G_0, \ell^2(G))$ has nontrivial kernel.
- We can pick a 1-cocycle $c : G \to \ell^2(G)$ with $c_h = 0$ for all $h \in G_0$.
- We can write $c_g(k) = F(g^{-1}k) - F(k)$.

  We may assume $F : G \to [1/3, 2/3]$.
- Define $X_0 = \{0, 1\}$ and $\mu_g(0) = F(g)$.

  Then, $G \curvearrowright \prod_{g \in G} (X_0, \mu_g)$ is ergodic and of type III_1.

Main idea: if a function F on $X \times \mathbb{R}$ is invariant under the Maharam extension, then it is G_0-invariant and thus, it only depends on the $\mathbb{R}$-variable.

Non orbit equivalent Bernoulli actions of free groups

- (Bowen) For a fixed n, all classical Bernoulli actions $\mathbb{F}_n \curvearrowright (X_0, \mu_0)^{\mathbb{F}_n}$ are orbit equivalent.

- (V-Wahl) There are many non-orbit equivalent, strongly ergodic, type III_1 Bernoulli actions $\mathbb{F}_n \curvearrowright \prod_{g \in \mathbb{F}_n} (X_0, \mu_g)$.

- When $G \curvearrowright (X, \mu)$ is nonsingular and strongly ergodic, then $H^1(G \curvearrowright X, S^1)$ is a Polish group.

  Denoting by $\omega \in H^1(G \curvearrowright X, \mathbb{R}^*_+)$ the Radon-Nikodym 1-cocycle, we obtain a group homomorphism $\mathbb{R} \to H^1(G \curvearrowright X, S^1) : t \mapsto \omega^{it}$.

  (Connes, Houdayer-Marrakchi-Verraedt) The τ-invariant is defined as the weakest topology on $\mathbb{R}$ making this homomorphism continuous. This is an orbit equivalence invariant.

- We obtain strongly ergodic, type III_1 Bernoulli actions $\mathbb{F}_n \curvearrowright \prod_{g \in \mathbb{F}_n} (X_0, \mu_g)$ with all kinds of prescribed τ-invariants.

Nonsingular Bernoulli actions of amenable groups

Let G be an infinite amenable group.

$\rightsquigarrow$ (Peterson-Thom) A 1-cocycle $c : G \to \ell^2(G)$ is either inner or proper.

So, if $c_g = 0$ on an infinite subgroup, then $c_g = 0$ for all $g \in G$.

Important step: when is $G \curvearrowright \prod_{g \in G} (\{0,1\}, \mu_g)$ conservative ?

Recall: a nonsingular action $G \curvearrowright (X, \mu)$ is **dissipative** if $X = \bigsqcup_{g \in G} g \cdot \mathcal{U}$, up to measure zero.

It is conservative if for every non-negligible $\mathcal{U} \subset X$, there exists a $g \neq e$ with $\mu(\mathcal{U} \cap g \cdot \mathcal{U}) > 0$.

Question: when is a nonsingular Bernoulli action conservative ?

Conservative Bernoulli actions

Consider a nonsingular Bernoulli action $G \curvearrowright \prod_{g \in G} (\{0,1\}, \mu_g)$.

- As before, write $F(g) = \mu_g(0)$.
- Denote $c : G \to \ell^2(G) : c_g(h) = F(g^{-1}h) - F(h)$.

Theorem (V-Wahl, 2017)

Assume $1/3 \leq F(g) \leq 2/3$ for all $g \in G$.

- If $\sum_{g \in G} \exp\left(-16 \, \|c_g\|_2^2\right) = +\infty$, the Bernoulli action is conservative.
- If $\sum_{g \in G} \exp\left(-\frac{1}{2} \, \|c_g\|_2^2\right) < \infty$, the Bernoulli action is dissipative.

Even for $G = \mathbb{Z}$, no conservativeness criterion was known before.

A multiple diagonal product may change conservative to dissipative.

Type III Bernoulli actions for amenable groups

Similar to Cornulier-Tessera-Valette : for G infinite amenable, there exist proper 1-cocycles $c : G \to \ell^2(G)$ of arbitrarily slow growth.

$\rightsquigarrow$ Plenty of conservative nonsingular Bernoulli actions.

Details in the case of $G = \mathbb{Z}$

- Fix $0 < \lambda < 1$ and $\mu_n(0) = \lambda$ for all $n \leq 0$.
- Assume that the conservativeness criterion holds (by choosing a slowly growing 1-cocycle).

If $\lim_{n \to +\infty} \mu_n(0) = \lambda$ and $\sum_n (\mu_n(0) - \lambda)^2 = +\infty$, then the Bernoulli action is ergodic and of type III_1.

Essence of the proof...

Faits divers and open questions

- For free groups $\mathbb{F}_n$ with $n \geq 2$ and for nontrivial free product groups $G = \Lambda * \mathbb{Z}$, we construct nonsingular Bernoulli actions of **type III_λ.**

- **Open problem:** does $\mathbb{Z}$ admit a Bernoulli action of type III_λ ?

  (Danilenko-Lemanczyk) The answer is no if we assume that the measures μ_n with $n \leq 0$ are the same.

- For the free group $\mathbb{F}_2$, we construct an explicit Bernoulli action $\mathbb{F}_2 \curvearrowright (X, \mu)$ with the following properties.
  - Ergodic and of type III_1.
  - The action of each individual element $g \in \mathbb{F}_2 \setminus \{e\}$ is dissipative.
  - The diagonal product $\mathbb{F}_2 \curvearrowright X^{220}$ is dissipative.

- **Open problem:** a concrete ergodicity criterion for nonsingular Bernoulli actions.