Exotic connected components of $\mathrm{SO}(p,q)$-Higgs bundle moduli

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Joint with
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Notation and background

- X — closed Riemann surface of genus $g \geqslant 2$.
- G — Connected real reductive Lie group.
- $H \subset G$ — maximal compact subgroup, Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

A G-Higgs bundle is (E, ϕ), where:

- $E \to X$ is a holomorphic principal $H^{\mathbb{C}}$-bundle;
- $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. Here $K = \Omega^1_X$ is the canonical bundle and $E(\mathfrak{m}^{\mathbb{C}}) = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$.

For $d \in \pi_1 H$,

$$M = M_d(X, G)$$

denotes the moduli space of G-Higgs bundles (E, ϕ) with topological type $c(E) = d$.

The basic problem

Question: What is $\pi_0(M_d(X,G))$?
If G is compact or complex, the answer is easy to state:

$$\pi_0(M_d(X,G)) = 1.$$

(Ramanathan, Li, García-Prada–Oliveira)

But for general real G this may **not** be the case. Known examples of disconnectedness of $M_d(X,G)$ arise in two different ways:

1. Hitchin components for split real G, and
2. The "*Cayley correspondence*" for maximal G-Higgs bundles for non-compact hermitian G of tube type (closely related to rigidity for maximal surface group representations).

$\mathrm{SO}_0(p,q)$-Higgs bundles

An $\mathrm{SO}_0(p,q)$-Higgs bundle (E,ϕ) corresponds to: (V,W,η), where

$$V, W \to X$$

are special orthogonal bundles (with quadratic forms Q_V and Q_W) of rank p and q, respectively, and

$$\eta\colon W \to V \otimes K.$$

The associated $\mathrm{GL}(p+q,\mathbb{C})$-Higgs bundle is

$$\left(V \oplus W, \begin{pmatrix} 0 & \eta \\ -\eta^* & 0 \end{pmatrix}\right).$$

Topological type: determined by

$$(w_2(V), w_2(W)) \in (\mathbb{Z}/2) \times (\mathbb{Z}/2)$$

(assuming $p,q \geqslant 3$).

Main Theorem

- Assume that: $q \geqslant p+2$, $p \geqslant 3$ and p odd.
- For $(a,b) \in (\mathbb{Z}/2) \times (\mathbb{Z}/2)$, let $M_{a,b} = M_{a,b}(X, \mathrm{SO}_0(p,q))$.

Theorem

For each $b = 0,1$, the moduli space $M_{(0,b)}$ has two connected components:

1. *A component $M^0_{(0,b)}$ which consists of those $\mathrm{SO}_0(p,q)$-Higgs bundles which can be deformed to $\mathrm{SO}(p) \times \mathrm{SO}(q)$-bundles;*
2. *An* **exotic** *component $M^e_{(0,b)}$.*

For each $b = 0,1$, the moduli space $M_{(1,b)}(X, \mathrm{SO}_0(p,q))$ is connected.

Remark

The case of p even is similar but details have to be double checked.

Further remarks

- This result does **not** fall into the previously known types of “extra” connected components already mentioned.
- The exotic components contain “positive representations” (Anna Wienhard’s talk yesterday).
- When $p = 2$, the group $\mathrm{SO}_0(p, q)$ is of hermitian type. This case was studied by Bradlow–García-Prada–G.
- The case $p = 1$ was studied by Aparicio–García-Prada.
- For $q = p, p + 1$, the group $\mathrm{SO}_0(p, q)$ is split real, and so there is a Hitchin component. For $q = p$, this is the only extra component. For $q = p + 1$, further extra components were found by Collier in his PhD thesis — see the previous talk!

Strategy of proof

The standard approach is to view the moduli space as the moduli space of solutions to Hitchin's equations and use Hitchin's proper functional:

$$f\colon M_d(X, \mathrm{SO}_0(p,q)) \to \mathbb{R},$$
$$(A, \phi) \mapsto \|\phi\|^2.$$

If the moduli space is smooth, then f is a moment map for the S^1-action and hence a perfect Bott–Morse function.
Even in the singular case, if the subspace of local minima of f on any subspace $X \subset M_d$ is connected, then X itself is connected.

Problem: we can distinguish the subspaces of local minima but we do not have a way of distinguishing the subspaces M^0 and M^e (as with the Cayley correspondence) or a parametrization (as for the Hitchin component).

Solution: Following Simpson, we use the $\mathbb{C}^*$-action. Additionally we need to analyse connecting orbits!

Projective embedding

Following Simpson, we may assume that

$$M \subset \mathbb{P}(V),$$

where the $\mathbb{C}^*$-action lifts to $V = \bigoplus V_l$ with

$$t \cdot v = t^l v$$

for $v \in V_l$.

Remark

This actually requires working with the image of M in the $\mathrm{GL}(p+q,\mathbb{C})$-Higgs bundle moduli space.

Limits and stratifications

Let $X \subset \mathbb{P}(V)$ be projective and write X_λ with $\lambda \in \Lambda$ for the components of the fixed locus of the $\mathbb{C}^*$-action.
The limits

$$x_0 = \lim_{t \to 0} t \cdot x \quad \text{and} \quad x_\infty = \lim_{t \to \infty} t \cdot x$$

exist and belong to the fixed locus. Moreover, the $\mathbb{C}^*$-orbit of x extends to

$$\mathbb{C}^* \subset \mathbb{P}^1 \to X.$$

The image of $\mathbb{P}^1$ is called an *orbit closure*. This gives the Białynicki-Birula stratifications, with strata:

$$X_\lambda^+ = \{x \in X \mid x_0 \in X_\lambda\},$$
$$X_\lambda^- = \{x \in X \mid x_\infty \in X_\lambda\}.$$

Partial ordering of the fixed loci

A connected component X_λ is *directly less* than a connected component X_μ if there exists a non-fixed x such that

$$x_0 \in X_\lambda \quad \text{and} \quad x_\infty \in X_\mu .$$

Let "$<$" denote the partial ordering induced by this relation.

The following result is due to Białynicki-Birula–Sommese (in the setting of normal complex analytic spaces). The proof involves a careful study of degenerations of orbit closures.

Theorem

Assume that we have a subset $\Lambda_1 \subset \Lambda$ with the following property:

- *If $\lambda \in \Lambda_1$ and $X_\lambda < X_\mu$ then $\mu \in \Lambda_1$.*

Then $\bigcup_{\lambda \in \Lambda_1} X_\lambda^+$ is closed in X.

Dealing with non-compactness

Problem:

For the Higgs bundle moduli space M only $\lim_{t\to 0}(E, t\phi)$ exists and there is a Białynicki-Birula stratification only by the M_λ^+. (The M_λ^- give a stratification of the nilpotent cone.)

Solution:

In order to apply the theory to Higgs bundle moduli we use the compactification of Hausel/Schmitt/Simpson obtained by adding in the missing limits $\lim_{t\to\infty}(E, t\phi)$, which exist in $\mathbb{P}(V)$.

The Białynicki-Birula–Sommese Theorem remains valid as stated for $X \subset \mathbb{P}(V)$ as long as $x_0 = \lim_{t\to 0} t \cdot x$ exists for all $x \in X$ and there is a proper $\mathbb{C}^*$-equivariant map:

$$h \colon X \to B = \bigoplus B_m,$$

where $\mathbb{C}^*$ acts on each weight space B_m with positive weight m.

Orbit closures and deformations

Recall that $V = \bigoplus V_l$, where $\mathbb{C}^*$ acts on V_l with weight l.

Let $x_0 \in \mathbb{P}(V_l)$ be fixed. Then there is a weight decomposition

$$T_{x_0}\mathbb{P}(V) = \bigoplus_w T_{x_0}\mathbb{P}(V)_w$$

(in fact $T_{x_0}\mathbb{P}(V)_w \simeq \bigoplus_{k-l=w} V_k$).

Assume that $l < k$ and let $x \in \mathbb{P}(V)$ be such that $x_0 = \lim_{t\to 0} t \cdot x$ and $x_\infty \in V_k$. Then the $(k-l)$-jet at 0 of the orbit closure $\mathbb{P}^1 \to \mathbb{P}(V)$ lies in $T_{x_0}\mathbb{P}(V)_{k-l}$, and one checks that this gives an identification:

$$\{\text{orbit closures connecting } \mathbb{P}(V_k) \text{ to } x_0\} \simeq T_{x_0}\mathbb{P}(V)_{k-l}. \qquad (*)$$

All this restricts to $X \subset \mathbb{P}(V)$ (except that not all infinitesimal deformations at a singular $x \in X$ are guaranteed to integrate).

Application to $M = M_d(X, \mathrm{SO}_0(p,q))$

Strategy:

1. Identify *minimal components* of $M^{\mathbb{C}^*}$: those which contain no limits of negative weight orbit closures.
2. Minimal components correspond to local minima of f. In fact, one may use a lemma of Simpson (instead of the topological argument involving the properness of f) to conclude connectedness of the whole space from connectedness of the minimal component.
3. Analyze the graph whose vertices are fixed components of the $\mathbb{C}^*$-action and whose edges correspond to orbit closures connecting these.
4. Show that the components of the graph are indexed by the minimal components, and conclude from Białynicki-Birula–Sommese that the same is true for the components of M.

Minimal fixed loci

A $\mathrm{SO}_0(p,q)$-Higgs bundle (V,W,η) is called *minimal* if it is fixed by $\mathbb{C}^*$ and there is no (V',W',η'), non-isomorphic to (V,W,η), and such that

$$\lim_{t\to\infty}(V',W',t\eta') = (V,W,\eta).$$

Remark

Minimal $\mathrm{SO}_0(p,q)$-Higgs bundles are exactly those which are local minima of the Hitchin functional f.

The classification of minimal $\mathrm{SO}_0(p,q)$-Higgs bundles in M^s (the smooth locus) was carried out in:

M. Aparacio Arroyo: *The Geometry of* $\mathrm{SO}(p,q)$*-Higgs Bundles*, PhD thesis, Univ. Salamanca, 2009,

and the following was conjectured:

Minimal fixed loci – 2

Recall that we assume $q \geqslant p+2$, $p \geqslant 3$ and p odd.

Proposition

A polystable $\mathrm{SO}_0(p,q)$-Higgs bundle (V,W,η) is minimal if and only if either $\eta = 0$ or (V,W,η) is **exotic**, *i.e., of the form*

$$K^{p-1} \xrightarrow{-1} K^{p-2} \xrightarrow{1} \cdots \xrightarrow{-1} K \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{-1} K^{-p+2} \xrightarrow{1} K^{-p+1}.$$
$$W_0$$

Here the maps ± 1 come from the identification $K^j \simeq K^{j-1} \otimes K$ and

- *even powers of K are subbundles of V and odd powers of K are subbundles of W;*
- *W_0 is a polystable $\mathrm{SO}(q-p+1,\mathbb{C})$-bundle.*

Important observation: Both the minimal locus $\eta = 0$ and the exotic minimal locus are connected.

Consider **exotic** fixed points for $\mathbb{C}^*$. By this we mean those of the form:

$$W_{-p} \xrightarrow{\eta_{-p}} K^{p-1} \xrightarrow{-1} \cdots \xrightarrow{-1} K \xrightarrow{1} \underset{W_0}{\mathcal{O}} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{-p+1} \xrightarrow{-\eta^*_{-p}} W_p$$

Here:

- even powers of K are subbundles of V and odd powers of K are subbundles of W;
- W_0 is a polystable $\mathrm{SO}(q-p-1,\mathbb{C})$-bundle;
- $W_{-p} \simeq W_p^*$ is a line bundle of degree d with $0 \leqslant d \leqslant p(2g-2)$.

Remark

If $d=0$ polystability forces $\eta_{-p}=0$ and the fixed point is a (strictly polystable) exotic minimal $\mathrm{SO}_0(p,q)$-Higgs bundle.

Main Lemma

For each $0 \leqslant d \leqslant p(2g-2)$, denote by $\mathcal{F}_d \subset M$ the locus of exotic fixed points with $\deg(W_{-p}) = d$.

Observation: $\mathcal{F}_d$ is connected.

Lemma

1. *Let $x \in M$ be such that $\lim_{t\to\infty} t \cdot x$ is exotic. Then $\lim_{t\to 0} t \cdot x$ is exotic.*
2. *Let $x \in M$ be such that $\lim_{t\to 0} t \cdot x$ is exotic. If $\lim_{t\to 0} t \cdot x$ exists, then it is exotic.*

Proof of the theorem

Denote by $\{M_\lambda\}_{\lambda\in\Lambda}$ the connected components of the fixed locus $M^{\mathbb{C}^*}$.

- Let $\Lambda_e \subset \Lambda$ consist of those λ for which M_λ is exotic (i.e., $M_\lambda = \mathcal{F}_d$ for some d).
- Let $\Lambda_0 = \Lambda \setminus \Lambda_e$.

Then the Main Lemma implies that both Λ_e and Λ_0 satisfy the hypothesis of the Białynicki-Birula–Sommese Theorem. Let

$$M^0 = \bigcup_{\lambda\in\Lambda_0} M_\lambda \quad \text{and} \quad M^e = \bigcup_{\lambda\in\Lambda_e} M_\lambda.$$

We conclude that $M = M^0 \cup M^e$ is a decomposition of M into disjoint closed non-empty connected subspaces. □

Infinitesimal deformations and Hodge bundles

The infinitesimal deformation space of (V, W, η) is $\mathbb{H}^1(C^\bullet(V, W, \eta))$, where

$$C^\bullet(V, W, \eta)\colon \Lambda^2 V \oplus \Lambda^2 W \xrightarrow{\mathrm{ad}(\eta)} \mathrm{Hom}(W, V) \otimes K$$

with $\mathrm{ad}(\eta)(a, b) = \eta a - b\eta$.

A $\mathrm{SO}_0(p, q)$-Higgs bundle (V, W, η) is fixed under $\mathbb{C}^*$ if and only if it is a **Hodge bundle**:

$$V = \bigoplus V_r, \quad W = \bigoplus W_s, \quad \eta\colon W_s \to V_{s+1} \otimes K.$$

In this case $C^\bullet = C^\bullet(V, W, \eta)$ decomposes accordingly as $C^\bullet = \bigoplus_w C^\bullet_w$ and

$$\mathbb{H}^1(C^\bullet)_w = \mathbb{H}^1(C^\bullet_{-w}).$$

Strategy of proof

Let (V, W, η) be an exotic fixed point. In view of the correspondence (*) between orbit closures and infinitesimal deformations, the following steps will provide the proof of the lemma:

Downward deformations:

1. for each negative direction in $\mathbb{H}^1(C^\bullet(V, W, \eta))$ identify an orbit $\mathbb{C}^* \cdot x$ with $x_\infty = (V, W, \eta)$;
2. find x_0 and show that it is exotic.

Upward deformations:

1. for each positive direction in $\mathbb{H}^1(C^\bullet(V, W, \eta))$ identify an orbit $\mathbb{C}^* \cdot x$ with $x_0 = (V, W, \eta)$;
2. find x_∞ **if it exists** and show that it is exotic.

Infinitesimal downward deformations

Recall that exotic fixed points are $\mathrm{SO}_0(p,q)$-Higgs bundles of the form

$$W_{-p} \xrightarrow{\eta_{-p}} K^{p-1} \xrightarrow{-1} \cdots \xrightarrow{-1} K \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{-p+1} \xrightarrow{-\eta_{-p}^*} W_p$$
$$W_0$$

Proposition

Let (V, W, η) *be an exotic fixed point. Then*

$$\mathbb{H}^1(C^\bullet(V,W,\eta))_{<0} = \mathbb{H}^1(C_p^\bullet) \simeq H^1(\mathrm{Hom}(W_0, W_p)).$$

Proof.

Easy check using the particular shape of (V, W, η). □

Integrating downward deformations

Let (V, W, η) be an exotic fixed point and let $s \in H^1(\mathrm{Hom}(W_0, W_p))$. Then s defines a deformation W_s of $W_{-p} \oplus W \oplus W_p$:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & W_p & \longrightarrow & W_s' & \longrightarrow & W_0 & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & W_s & & & & \\ & & & & \downarrow & & & & \\ & & & & W_{-p} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0. & & & & \end{array}$$

Integrating downward deformations – 2

Defining η_s in the obvious way gives a deformation (V, W_s, η_s):

$$K^{p-1} \xrightarrow{-1} \cdots \xrightarrow{-1} K \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{-p+1}.$$
$$W_s \xrightarrow{\tilde{\eta}_s} K^{p-1}, \qquad K^{-p+1} \xrightarrow{-\tilde{\eta}_s^*} W_s$$

Proposition

The $\mathrm{SO}_0(p,q)$-Higgs bundle (V, W_s, η_s) is polystable. Moreover,

- $\lim_{t\to\infty}(V, W_s, t\eta_s) = (V, W, \eta)$, *and*
- *the p-jet at $\infty \in \mathbb{P}^1$ of its orbit closure is* $s \in H^1(\mathrm{Hom}(W_0, W_p)) = \mathbb{H}^1(C^\bullet(V, W, \eta))_{-p}$.

The downward limit of a downward deformation

What is $\lim_{t\to 0}(V, W_s, t\eta_s)$?

If W_s is polystable, the limit is simply the exotic minimal $\mathrm{SO}_0(p,q)$-Higgs bundle

$$K^{p-1} \xrightarrow{-1} K^{p-2} \xrightarrow{1} \cdots \xrightarrow{-1} K \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{-1} K^{-p+2} \xrightarrow{1} K^{-p+1}.$$

$$W_s$$

Otherwise, we have the following:

Proposition

The maximal destabilizing $F \subset W_s$ is an isotropic line bundle with $0 < \deg(F) < \deg(W_{-p})$.

The downward limit of a downward deformation – 2

Define an $\mathrm{SO}(q-p+1,\mathbb{C})$-bundle W_0' and a map $\alpha\colon F \to K^{p-1} \otimes K$ by:

$$0 \to F^{\perp} \to W_s \to F^* \to 0$$
$$0 \to F \to F^{\perp} \to W_0' \to 0$$
$$\alpha := \tilde{\eta}_{s|F}.$$

Proposition

If W_s is unstable, the limit $\lim_{t\to 0}(V, W_s, t\eta_s)$ is the exotic fixed point

$$F \xrightarrow{\alpha} K^{p-1} \xrightarrow{-1} \cdots \xrightarrow{-1} K \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{-p+1} \xrightarrow{-\alpha^*} F^{-1}.$$
$$W_0'$$

Upward deformations

The analysis is similar in spirit but complicated by the fact that we must distinguish orbits which do not have a limit as $t \to \infty$ and orbits in the nilpotent cone.

Proposition

Let (V, W, η) be an exotic fixed point. Then the infinitesimal deformation corresponding to an orbit in the nilpotent cone lies in $\mathbb{H}^1(C^\bullet_{-p}) = \mathbb{H}^1(C^\bullet)_p$.

Any non-zero $s \in \mathbb{H}^1(C^\bullet)_p$ integrates to an $\mathrm{SO}_0(p, q)$-Higgs bundle of the form

$$K^{p-1} \xrightarrow{-1} \cdots \xrightarrow{-1} K^{-1} \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K \xrightarrow{1} \cdots \xrightarrow{1} K^{-p+1} \, .$$

$$W_s \xrightarrow{\tilde{\eta}_s} K^{p-1}, \qquad K^{-p+1} \xrightarrow{-\tilde{\eta}_s^*} W_s$$

(This is harder to show than the corresponding result for downwards deformations.)

Distinguishing nilpotent orbits

Let (V, W_s, η_s) be the deformation constructed from a non-zero $s \in \mathbb{H}^1(C^\bullet)_p$ in the previous slide. Define a subbundle $N \subset W_s$ by

$$N = \ker(\tilde{\eta}_s) \subset W_s.$$

Proposition

The orbit of (V, W_s, η_s) is contained in the nilpotent cone if and only if $N \subset W_s$ is co-isotropic.

Moreover, we have the following:

Proposition

The line bundle $F = W_s/N = (N^\perp)^*$ satisfies $\deg(W_{-p}) < \deg(F) \leqslant p(2g-2)$.

The upward limit of an upward deformation

Since $N = \ker(\tilde{\eta}_s)$, there is an induced map $\gamma\colon F \to K^{p-1} \otimes K$.

Moreover, we have an $\mathrm{SO}(q-p-1,\mathbb{C})$-bundle $W_0' = N/N^{\perp}$.

Proposition

When the orbit of (V, W_s, η_s) is contained in the nilpotent cone, its limit as $t \to \infty$ is the exotic fixed point

$$F \xrightarrow{\gamma} K^{p-1} \xrightarrow{-1} \cdots \xrightarrow{-1} K \xrightarrow{1} \mathcal{O} \xrightarrow{-1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{-p+1} \xrightarrow{-\gamma^*} F^{-1}.$$
$$W_0'$$

Note: Upward deformations from exotic minimal $\mathrm{SO}_0(p,q)$-Higgs bundles require separate but similar treatment.

Final Remark: Our analysis gives a fairly explicit description of all exotic $\mathrm{SO}_0(p,q)$-Higgs bundles.

Happy Birthday Nigel!