Simultaneously continuous retractions and Bishop-Phelps-Bollobás property

Han Ju Lee (Dongguk University — South Korea)

We study the existence of a retraction from the dual space X^* of a (real or complex) Banach space X onto its unit ball B_{X^*} which is uniformly continuous in norm topology and continuous in weak-$*$ topology. Such a retraction is called a uniformly simultaneously continuous retraction. It is shown that if X has a normalized unconditional Schauder basis with unconditional basis constant 1 and X^* is uniformly monotone, then a uniformly simultaneously continuous retraction from X^* onto B_{X^*} exists. It is also shown that if $\{X_i\}$ is a family of separable Banach spaces whose duals are uniformly convex with moduli of convexity $\delta_i(\epsilon)$ such that $\inf_i \delta_i(\epsilon) > 0$ and $X = [\bigoplus X_i]_{c_0}$ or $X = [\bigoplus X_i]_{\ell_p}$ for $1 \leq p < \infty$, then a uniformly simultaneously continuous retraction exists from X^* onto B_{X^*}.
The relation between the existence of a uniformly simultaneously continuous retraction and the Bishsop-Phelps-Bollobás property for operators is investigated and it is proved that the existence of a uniformly simultaneously continuous retraction from X^* onto its unit ball implies that a pair $(X, C_0(K))$ has the Bishop-Phelps-Bollobás property for every locally compact Hausdorff spaces K. As a corollary, we prove that $(C_0(S), C_0(K))$ has the Bishop-Phelps-Bollobás property if $C_0(S)$ and $C_0(K)$ are the spaces of all real-valued continuous functions vanishing at infinity on locally compact metric space S and locally compact Hausdorff space K respectively.