The fixed point property for unbounded sets

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Consider D is a subset of a metric space X. $T\colon D \to D$ is said to be nonexpansive mapping if it is a Lipschitzian mapping with Lispschitz constant at most 1. We say that D has the fixed point property for nonexpansive mappings if any such mapping T has at least one fixed point, that is, there exists $x \in D$ such that $Tx = x$.

A systematic study of this property began in mid 60's of the twentieth century when it turned out to be closely related to the geometry of the ambient space X and it is still active. Typically D is considered to be a *bounded* closed and convex subset of an infinite dimensional Banach space X. If the boundedness condition is removed from the set D then the problem drastically changes and different approaches are to be considered. One of the most relevant results in this sense was obtained in 1980 by W.O. Ray when he showed that a convex and closed subset of a real Hilbert space has the fixed point property if and only if it is bounded. For a very long time it has been unknown whether this was a charactering property for Hilbert spaces or, rather, if it would be a common property for any Banach space. It was in 2012 that T. Domínguez-Benavides found some other Banach spaces satisfying this property too.

Ray's property for Hilbert spaces has also been studied by different authors assuming that the ambient space X is not a Banach space but a geodesic metric space with a hyperbolic geometry. Examples of such spaces are, for instance, CAT(0) spaces from which Hilbert spaces are a particular case. These studies were initiated in the mid 80's and, in contrast to the normed case, examples of such spaces failing Ray's property were easily found.

In this mainly expository talk we present a review of this problem pointing out some of the most relevant facts, results and open problems about the same. A special emphasis will be put on some recent results showing the close relation between the failure of Ray's property and the hyperbolic geometry of the space.